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Constants of the Motion in Lagrangian Mechanics

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Abstract

Two points of view of the relationship between symmetries of a dynamical system and constants of the motion, in the Lagrangian framework, are compared. The first point of view is that associated with Noether's theorem, the second, with the Cartan form approach to Lagrangian mechanics. It is argued that the second is more satisfactory.

1. *Introduction*

My purpose in this paper is to compare two different views of the relationship between constants of the motion of a dynamical system with a finite number of degrees of freedom, and symmetries of the system. Both views assume the Lagrangian formulation of the equations of motion. The first is the well-known range of ideas associated with Noether's theorem. The second point of view is perhaps less familiar: It is based on E. Cartan's idea that the fundamental geometrical quantity, in the potentially very geometrical theory of Lagrangian mechanics, is not the Lagrangian function, but the exterior derivative of a certain differential one-form now known as the Cartan form.

I hope to show that Cartan's point of view is the superior one. It leads to a straightforward theory in which the notion of symmetry, and the relationship between symmetries and constants of the motion, are entirely natural. Noether's theorem, in this context, is neither so clear nor so universal; and contrary to the claims of several authors (Palmieri and Vitale, 1970; Saletan and Cromer, 1971), it does not appear to have a converse—as it is formulated here, at any rate.

To fully explain the differences between the two points of view it is desirable to use the techniques and terminology of modern differential geometry—tangent bundles, vector fields, differential forms, the Lie derivative, and so on. It is only by using this language that one can clearly express the necessary ideas about transformations and symmetries. The formulation of Lagrangian mechanics in these terms is quite commonplace now. R. Hermann's approach, in his book *Differential Geometry and the Calculus of Variations* (Hermann, 1968), is close in spirit to the approach adopted in this paper. I give a brief derivation of the expression of Lagrange's equations by means of the Cartan form in the first main section of the paper. However, it is obviously impossible in a short paper to derive all the necessary properties of vector fields and differential forms; instead, I shall merely list some of the main notations used below.

2. *Notation*

$\langle X, \omega \rangle$ denotes the pairing of a vector field X and a one-form ω, that is, the function whose value at any point x is the value of the covector ω_x on the vector X_x. $X \lrcorner \Omega$ is the contraction of the p-form Ω with the vector field X; if Ω is a two-form, for example, then $X \lrcorner \Omega$ is the one-form given by

$$\langle Y, X \lrcorner \Omega \rangle = \Omega(X, Y)$$

for any vector field Y. L_X denotes the Lie derivative along the vector field X. Frequent use is made of the formula

$$L_X \alpha = X \lrcorner d\alpha + d\langle X, \alpha \rangle$$

for the Lie derivative of a one-form α.

3. *The Cartan Form*

In Lagrangian mechanics one represents the dynamics of a mechanical system by a vector field on the tangent bundle to the space of configurations of the system (rather than one on the cotangent bundle, which is the domain of Hamiltonian mechanics). In fact the theory is best worked out in evolution space, $T(M) \times R$; M is the configuration space (a differentiable manifold), $T(M)$ its tangent bundle, and the real line R is added to allow one to deal with the time explicitly. It seems to be advantageous to include the time explicitly even when the Lagrangian is not directly dependent on time.

Since all considerations in this paper are of a purely local nature, it will frequently be possible to use coordinates. The coordinates of a point p in evolution space will be written $(x^1, x^2, \ldots, x^n, v^1, v^2, \ldots, v^n, t)$; here $(x^1, x^2, \ldots, x^n)$ are the coordinates of $\pi(p)$, the projection of p into M, with respect to some coordinate system in M; $(v^1, v^2, \ldots, v^n)$ are the components of a tangent vector to M at $\pi(p)$ with respect to the coordinate basis $\partial/\partial x^1, \partial/\partial x^2, \ldots, \partial/\partial x^n$; and t is the time. The system in question has n degrees of freedom.

The motions of the mechanical system are the projections into M of the integral curves of a certain vector field Γ on $T(M) \times R$. This vector field can be specified by first choosing a local basis for one-forms on $T(M) \times R$, and then giving the values of these one-forms on Γ. A suitable choice of one-forms is

$$\begin{aligned}
&dx^a - v^a dt, \qquad a = 1, 2, \ldots, n \\
&dv^a - \Lambda^a dt, \qquad a = 1, 2, \ldots, n \\
&dt
\end{aligned}$$

The functions $\Lambda^1, \Lambda^2, \ldots, \Lambda^n$ are given in terms of the Lagrangian function L of the system, which is a function on $T(M) \times R$, as follows:

$$\frac{\partial^2 L}{\partial v^a \partial v^b} \Lambda^b = \frac{\partial L}{\partial x^a} - \frac{\partial^2 L}{\partial v^a \partial x^b} v^b - \frac{\partial^2 L}{\partial v^a \partial t}$$

(it is assumed that the Lagrangian L is regular, in other words, that the matrix $\partial^2 L/\partial v^a \partial v^b$ is nonsingular). This choice of one-forms is appropriate because the definition of Γ is very simple:

$$\langle \Gamma, dx^a - v^a dt \rangle = 0$$

$$\langle \Gamma, dv^a - \Lambda^a dt \rangle = 0$$

$$\langle \Gamma, dt \rangle = 1$$

The first set of these conditions, and the last, have a straightforward geometrical significance. Any curve in M may be lifted to a curve in $T(M) \times R$ by adjoining to it its tangent vectors, and by identifying its parameter with the time. The tangent vector to any such lifted curve is annihilated by the one-forms $dx^a - v^a dt$, and gives 1 when paired with dt. The fact that Γ behaves in this manner shows, conversely, that its integral curves are obtained by lifting to $T(M) \times R$ curves in M in just the same way.

The conditions

$$\langle \Gamma, dv^a - \Lambda^a dt \rangle = 0$$

amount to Lagrange's equations, as must be evident from the definition of the functions Λ^a. Since $\langle \Gamma, dx^a - v^a dt \rangle = \langle \Gamma, dv^a - \Lambda^a dt \rangle = 0$, the contraction of Γ with any form constructed from the one-forms $dx^a - v^a dt$ and $dv^a - \Lambda^a dt$ by taking exterior products and multiplying by arbitrary functions must also vanish. In particular

$$\Gamma \lrcorner \left(\frac{\partial^2 L}{\partial x^a \partial V^b} (dx^a - v^a dt) + g_{ab}(dv^a - \Lambda^a dt) \right) \wedge (dx^b - v^b dt) = 0$$

(Here g_{ab} is written for $\partial^2 L/\partial v^a \partial v^b$, a convenient notation, and one which is appropriate since $\partial^2 L/\partial v^a \partial v^b$ plays a role in Lagrangian mechanics somewhat similar to the role played by the metric in Riemannian geometry, to which of course it reduces when L is actually the "kinetic energy" of a Riemannian metric.) This result about Γ is important for two reasons. In the first place it is equivalent to the conditions $\langle \Gamma, dx^a - v^a dt \rangle = \langle \Gamma, dv^a - \Lambda^a dt \rangle = 0$, not just a consequence of them: In other words, Γ is uniquely specified by the conditions

$$\Gamma \lrcorner \left(\frac{\partial^2 L}{\partial x^a \partial V^b} (dx^a - v^a dt) + g_{ab}(dv^a - \Lambda^a dt) \right) \wedge (dx^b - v^b dt) = 0$$

Second, this two-form is the exterior derivative of a one-form, which I shall denote by θ:

$$\theta = L dt + \frac{\partial L}{\partial v^a} (dx^a - v^a dt)$$

It is this one-form θ that is called the Cartan form. The conditions that determine Γ may be expressed more succinctly as

$$\Gamma \lrcorner\, d\theta = 0, \qquad \langle \Gamma, dt\rangle = 1$$

The first says that Γ is what is known as a characteristic vector field of $d\theta$.

The analysis so far has been based on the fact that $dx^a - v^a dt$, $dv^a - \Lambda^a dt$, dt form a basis for the one-forms on $T(M) \times R$. Locally at least, each one-form on $T(M) \times R$ can be expressed uniquely in terms of these. (In the case of θ, it is interesting to note that there is no dependence on $dv^a - \Lambda^a dt$.) The coefficients in the expression of a one-form α are easily determined. In fact

$$\alpha = \left\langle \frac{\partial}{\partial x^a}, \alpha \right\rangle (dx^a - v^a dt) + \left\langle \frac{\partial}{\partial v^a}, \alpha \right\rangle (dv^a - \Lambda^a dt) + \langle \Gamma, \alpha\rangle\, dt$$

In particular, if $\alpha = df$ is exact, one obtains

$$df = \frac{\partial f}{\partial x^a}(dx^a - v^a dt) + \frac{\partial f}{\partial v^a}(dv^a - \Lambda^a dt) + \Gamma(f)\, dt$$

and if f should be independent of v^a, then

$$df = \frac{\partial f}{\partial x^a}(dx^a - v^a dt) + \left(v^a \frac{\partial f}{\partial x^a} + \frac{\partial f}{\partial t}\right) dt$$

Since Γ is determined in terms of $d\theta$ (rather than θ itself), modifications to θ that do not affect $d\theta$ will not change Γ either. If θ and θ' are Cartan forms (for different Lagrangians) that have the same exterior derivative, and so lead to the same equations of motion, they must differ by a closed form, and so locally there is some function f such that

$$\theta' = \theta + df$$

Since both θ and θ' are independent of $dv^a - \Lambda^a dt$, so also must df be, and thus f must be independent of v^a. Then

$$\theta' = \left[L + \left(v^a \frac{\partial f}{\partial x^a} + \frac{\partial f}{\partial t}\right)\right] dt + \left(\frac{\partial L}{\partial v^a} + \frac{\partial f}{\partial x^a}\right)(dx^a - v^a dt)$$

and corresponds to the Lagrangian obtained by adding to L the "total time derivative" of f (to use a classical phrase).

4. *Constants of the Motion—the Cartan Point of View*

A constant of the motion for a dynamical system is a function on $T(M) \times R$ that is constant along each integral curve of Γ; in other words, a function F for which

$$\Gamma(F) = 0$$

But $\Gamma(F)$ is the coefficient of dt in the expression for dF in terms of $dx^a - v^a dt$, $dv^a - \Lambda^a dt$, dt. So another way of saying that F is a constant of the motion is to say that dF, when expressed in terms of that basis, does not involve dt. Then the problem of finding constants of the motion amounts to the problem of finding exact one-forms amongst the collection of one-forms generated by $dx^a - v^a dt$ and $dv^a - \Lambda^a dt$.

The exterior derivative of the Cartan form contains, in the one object, all the information contained by the $2n$ one-forms $dx^a - v^a dt$ and $dv^a - \Lambda^a dt$. It is thus natural to try to reinterpret what was said above about constants of the motion in terms of $d\theta$. This leads in a natural way to a relation between constants of the motion and symmetries of the system.

The situation is this. To every vector field X on $T(M) \times R$ there corresponds a one-form $X \lrcorner\ d\theta$, which is a linear combination of the one-forms $dx^a - v^a dt$ and $dv^a - \Lambda^a dt$; and conversely to every such one-form there corresponds a vector field—or rather, to be precise, a collection of vector fields. This correspondence can easily be found explicitly; in fact

$$\xi^a \frac{\partial}{\partial x^a} + \eta^a \frac{\partial}{\partial v^a} + \tau \frac{\partial}{\partial t} \to -g_{ab}(\xi^b - v^b \tau)(dv^a - \Lambda^a dt) + \textit{terms in } (dx^a - v^a dt)$$

A more elegant treatment uses the fact that for each point p in evolution space E, the map

$$T_p(E) \to T_p^*(E)$$

by

$$\xi \to \xi \lrcorner\ d\theta_p$$

is a linear map whose kernel is the one-dimensional subspace of $T_p(E)$ consisting of multiples of Γ_p. The image space is therefore $2n$ dimensional. If α lies in this image space, then there is some $\xi \in T_p(E)$ such that

$$\xi \lrcorner\ d\theta_p = \alpha$$

But then

$$\begin{aligned} \langle \Gamma_p, \alpha \rangle &= \langle \Gamma_p, \xi \lrcorner\ d\theta_p \rangle \\ &= d\theta_p(\xi, \Gamma_p) \\ &= 0 \end{aligned}$$

since Γ is characteristic for $d\theta$. Thus the image space of $T_p(E)$ under this map is certainly contained in the subspace of $T_p^*(E)$ consisting of those one-forms that are linear combinations of $dx^a - v^a dt$ and $dv^a - \Lambda^a dt$—which is just the subspace of those α such that $\langle \Gamma_p, \alpha \rangle = 0$. Since both spaces are $2n$ dimensional, they must coincide. In other words, $\langle \Gamma_p, \alpha \rangle = 0$ if and only if there is a vector $\xi \in T_p(E)$ such that $\alpha = \xi \lrcorner\ d\theta_p$.

To every one-form that gives zero when paired with Γ, there corresponds a vector field; two vector fields yield the same one-form if and only if they differ by a multiple of Γ.

To find a constant of the motion one has to find a one-form that gives zero when paired with Γ and that is exact. What condition does this latter requirement impose on a corresponding vector field X? If $X \lrcorner\ d\theta$ is exact, then so is $X \lrcorner\ d\theta + d\langle X, \theta\rangle$; but

$$X \lrcorner\ d\theta + d\langle X, \theta\rangle = L_X\theta$$

the Lie derivative of θ. And if $L_X\theta$ is exact, then

$$L_X d\theta = 0$$

It is natural to describe a vector field X, which has the property that the Lie derivative along X of the principle geometrical object of a system vanishes, as a symmetry of the system. The one-parameter group of transformations generated by X will pull $d\theta$ back to itself, after all. In this sense (and ignoring difficulties about closure of forms not necessarily implying exactness), constants of the motion correspond to symmetries, via the Cartan form. To be precise, we have the following:

Theorem.

(i) If the vector field X is such that $L_X\theta$ is exact, say $L_X\theta = df$, then $f - \langle X, \theta\rangle$ is a constant of the motion.

(ii) If F is a constant of the motion, there is a vector field X such that $L_X\theta = d(F + \langle X, \theta\rangle)$, and any vector field $X + g\Gamma$, where g is any function on $T(M) \times R$, has the same property.

(iii) Of all the vector fields (differing by multiples of Γ) which generate a given constant of the motion, there is a unique one Y such that $\langle Y, dt\rangle = 0$; this vector field Y satisfies $[Y, \Gamma] = 0$, so the corresponding one-parameter group of transformations permutes the integral curves of Γ.

Proof. (i) If $L_X\theta = df$, then

$$\begin{aligned} X \lrcorner\ d\theta &= L_X\theta - d\langle X, \theta\rangle \\ &= d(f - \langle X, \theta\rangle) \end{aligned}$$

and so

$$\Gamma(f - \langle X, \theta\rangle) = \langle \Gamma, (X \lrcorner\ d\theta)\rangle = 0$$

and $f - \langle X, \theta\rangle$ is a constant of the motion.

(ii) Conversely, suppose that F is a constant of the motion; then $\langle \Gamma, dF\rangle = 0$, so there is at least one vector field X such that

$$X \lrcorner\ d\theta = dF$$

Then

$$\begin{aligned} L_X\theta &= X \lrcorner\ d\theta + d\langle X, \theta\rangle \\ &= d(F + \langle X, \theta\rangle) \end{aligned}$$

Moreover, $X + g\Gamma = Y$ also satisfies $Y \lrcorner\, d\theta = dF$ (whatever the function g may be), and so by the same argument, $L_Y\theta = d(F + \langle Y, \theta\rangle)$.

(iii) If X generates a constant of the motion, and $\langle X, dt\rangle \neq 0$, then $X - \langle X, dt\rangle\Gamma = Y$ generates the same constant of the motion, and $\langle Y, dt\rangle = 0$. Now since $L_Y d\theta = 0$ and $\Gamma \lrcorner\, d\theta = 0$, it follows that

$$
\begin{aligned}
0 &= L_Y(\Gamma \lrcorner\, d\theta) \\
&= [Y, \Gamma] \lrcorner\, d\theta + \Gamma \lrcorner\, L_Y d\theta \\
&= [Y, \Gamma] \lrcorner\, d\theta
\end{aligned}
$$

Thus $[Y, \Gamma]$ is a characteristic vector field for θ, and so must be a multiple of Γ. (This is true for any vector field that is a symmetry, whether it satisfies $\langle X, dt\rangle = 0$ or not; in general such X will permute the integral curves of Γ as curves, but will change their parametrization.) Since $\langle Y, dt\rangle = 0$,

$$
\begin{aligned}
\langle [Y, \Gamma], dt\rangle &= [Y, \Gamma](t) \\
&= Y(\Gamma(t)) - \Gamma(Y(t)) \\
&= Y\langle \Gamma, dt\rangle - \Gamma\langle Y, dt\rangle \\
&= Y(1) - \Gamma(0) \\
&= 0
\end{aligned}
$$

But $\langle \Gamma, dt\rangle = 1$, so $[Y, \Gamma]$ can be only the zero multiple of Γ; in other words,

$$[Y, \Gamma] = 0$$

Thus the one-parameter group of transformations of $T(M) \times R$ generated by Y permutes integral curves of Γ, or maps motions to motions. □

Note that if the symmetry vector field is

$$X = \xi^a \frac{\partial}{\partial x^a} + \eta^a \frac{\partial}{\partial v^a} + \tau \frac{\partial}{\partial t}$$

then the constant of the motion is

$$f - \langle X, \theta\rangle = f - \left[L\tau + \frac{\partial L}{\partial v^a}(\xi^a - v^a\tau) \right]$$

In terms of the Cartan form, therefore, one has a very clear picture of the relationship between symmetries and constants of the motion. A symmetry is a vector field X such that $L_X d\theta = 0$; then locally at least, $X \lrcorner\, d\theta$ is exact, and the function whose exterior derivative it is, is a constant of the motion; moreover, X may be normalized so that it permutes the integral curves of Γ.

5. *Noether's Theorem*

The notion of symmetry in Noether's theorem is expressed in terms of the Lagrangian, rather than the Cartan form. In its simplest version, the procedure is to consider the effect of a transformation of configuration space (that is, a point transformation) on the Lagrangian. But since the Lagrangian is a function of the velocity variables, and possibly also the time—in other words, a function on evolution space—it is necessary to extend the transformation to a transformation of evolution space. More generally, to encompass the conservation of energy when the Lagrangian is time independent, for example, one may consider transformations of configuration space and time. There is then an additional complication: As well as finding how to extend the transformation to evolution space, one has to consider the invariance not of the Lagrangian L itself, but of the one-form Ldt.

I look first at the question of extending transformations of $M \times R$ to transformations of $T(M) \times R$.

I shall call the set of one-forms $dx^a - v^a dt$, and those one-forms that are linear combinations of them (with functions as coefficients) the contact one-forms, and denote them by C. The tangent vector of any curve in $T(M) \times R$ that is obtained from a curve in M, parametrized by time, by adjoining its tangent vectors, is annihilated by all the contact forms. This is their geometrical significance. The extension of a vector field on $M \times R$ to $T(M) \times R$ is achieved by demanding that the extended vector field preserve contact forms.

Proposition. Let Z be a vector field on $M \times R$. There is a unique vector field X on $T(M) \times R$ that projects onto Z and preserves contact forms, in the sense that $L_X(dx^a - v^a dt) \in C$, for all $a = 1, 2, \ldots, n$.

Proof. Suppose

$$Z = \xi^a \frac{\partial}{\partial x^a} + \tau \frac{\partial}{\partial t}$$

where ξ^a and τ are functions of x^a and t. Then

$$X = \xi^a \frac{\partial}{\partial x^a} + \eta^a \frac{\partial}{\partial v^a} + \tau \frac{\partial}{\partial t}$$

and the problem is to determine the functions η^a. Now

$$\begin{aligned} L_X(dx^a - v^a dt) &= d\xi^a - \eta^a dt - v^a d\tau \\ &= \left(\frac{\partial \xi^a}{\partial x^b} - v^a \frac{\partial \tau}{\partial x^b}\right)(dx^b - v^b d\tau) \\ &\quad + \left[v^b \frac{\partial \xi^a}{\partial x^b} + \frac{\partial \xi^a}{\partial t} - v^a\left(v^b \frac{\partial \tau}{\partial x^b} + \frac{\partial \tau}{\partial t}\right) - \eta^a\right] dt \end{aligned}$$

So $L_X(dx^a - v^a dt) \in C$ if and only if

$$\eta^a = v^b \frac{\partial \xi^a}{\partial x^b} + \frac{\partial \xi^a}{\partial t} - v^a \left(v^b \frac{\partial \tau}{\partial x^b} + \frac{\partial \tau}{\partial t} \right)$$
$$= \dot{\xi}^a - v^a \dot{\tau}$$

to use an obvious classical notation. So the functions η^a are determined by the condition $L_X(dx^a - v^a dt) \in C$. □

The most general version of Noether's theorem usually presented (Lovelock and Rund, 1975, for example) is the following.

Theorem. If X preserves contact forms, and projects onto a vector field in $M \times R$, and if there is a function f on $T(M) \times R$ such that

$$L_X(Ldt) - df \in C$$

then $f - \langle X, \theta \rangle$ is a constant of the motion, where θ is the Cartan form

Proof.

$$L_X\theta = L_X \left[Ldt + \frac{\partial L}{\partial v^a} (dx^a - v^a dt) \right]$$

$$= L_X(Ldt) + X \left(\frac{\partial L}{\partial v^a} \right) (dx^a - v^a dt) + \frac{\partial L}{\partial v^a} L_X(dx^a - v^a dt)$$

$$= df + \alpha \qquad \text{where } \alpha \in C$$

Thus

$$X \lrcorner\, d\theta = d(f - \langle X, \theta \rangle + \alpha$$

Now, $\langle \Gamma, \alpha \rangle = 0$ since $\alpha \in C$. Thus $\Gamma(f - \langle X, \theta \rangle) = 0$, and so $f - \langle X, \theta \rangle$ is a constant of the motion. □

There are several points worth noting about this proof. First, the hypothesis that X projects onto a vector field on $M \times R$ is not in fact used in the proof. Thus there is a generalization of Noether's theorem to vector fields on $T(M) \times R$ that preserve contact forms, but that do not necessarily come from extending vector fields on $M \times R$. Second, the Cartan form plays an important role in both theorem and proof–in the specification of the actual constant of the motion, and in the way the defining property of Γ as characteristic vector field for $d\theta$ is used. Third, despite this, X is not necessarily a symmetry in the earlier sense–it is not necessarily the case that $L_X d\theta = 0$. Fourth, the equation

$$X \lrcorner\, d\theta = d(f - \langle X, \theta \rangle) + \alpha$$

is, despite appearances, essentially equivalent to the classical fundamental variational formula. In fact, if

$$X = \xi^a \frac{\partial}{\partial x^a} + \eta^a \frac{\partial}{\partial v^a} + \tau \frac{\partial}{\partial t}$$

and if

$$t \to x^a(t)$$

is a curve in M, which lifts to the curve

$$t \to (x^a(t), \dot{x}^a(t), t)$$

in $T(M) \times R$ (not necessarily an integral curve of Γ), with tangent vector

$$T = \dot{x}^a \frac{\partial}{\partial x^a} + \ddot{x}^a \frac{\partial}{\partial v^a} + \frac{\partial}{\partial t}$$

then

$$\begin{aligned}\langle T, X \lrcorner\, d\theta\rangle &= -\langle T, g_{ab}(\xi^b - v^b\tau)(dv^a - \Lambda^a dt)\rangle \\ &= -g_{ab}(\xi^b - v^b\tau)(\ddot{x}^a - \Lambda^a) \\ &= \left[\frac{\partial L}{\partial x^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^a}\right)\right](\xi^a - v^a\tau)\end{aligned}$$

Use has been made here of the fact that

$$\langle T, dx^a - v^a dt\rangle = 0$$

the terms $\partial L/\partial x^a$ and $\partial L/\partial v^a$ are to be regarded here as functions on the curve, and d/dt denotes differentiation along the curve. Moreover,

$$\langle T, d(f - \langle X, \theta\rangle) + \alpha\rangle = \frac{d}{dt}\left(f - \left[L\tau + \frac{\partial L}{\partial v^a}(\xi^a - v^a\tau)\right]\right)$$

Thus the fundamental variational formula is obtained:

$$\left(\frac{\partial L}{\partial x^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^a}\right)\right)(\xi^a - v^a\tau) = \frac{d}{dt}\left\{f - \left[L\tau + \frac{\partial L}{\partial v^a}(\xi^a - v^a\tau)\right]\right\}$$

Judging from its hypotheses, Noether's theorem has little to do with the earlier result involving the Cartan form. Actually, the two are quite closely related. The crucial point of the proof of Noether's theorem is that

$$L_X\theta = df + \alpha$$

where the one-form α is such that $\langle\Gamma, \alpha\rangle = 0$. Now since $\langle\Gamma, \alpha\rangle = 0$, there is a vector field Y such that

$$Y \lrcorner\, d\theta = \alpha$$

Consider the vector field $Z = X - Y$. Then

$$\begin{aligned} L_Z\theta &= L_X\theta - L_Y\theta \\ &= df + \alpha - (Y \lrcorner\, d\theta + d\langle Y, \theta\rangle) \\ &= d(f - \langle Y, \theta\rangle) \end{aligned}$$

Thus $L_Z\theta$ is exact; Z satisfies the hypothesis of the Cartan form result, and so generates a constant of the motion according to that theorem, namely,

$$(f - \langle Y, \theta\rangle) - \langle Z, \theta\rangle = f - \langle X, \theta\rangle$$

In fact Z generates the same constant of the motion, via the Cartan form, as X does via Noether's theorem.

The possibility of a converse to Noether's theorem has been discussed (Palmieri and Vitale, 1970; Saletan and Cromer, 1971)—given a constant of the motion, can one reconstruct a vector field that generates it according to Noether's theorem? I shall show that even in the generalized form noted above, the theorem has no converse.

First, I examine the condition that a vector field X on $T(M) \times R$, which does not necessarily project onto a vector field on $M \times R$, preserves contact forms. Let $X = \xi^a\, \partial/\partial x^a + \eta^a\, \partial/\partial v^a + \tau\, \partial/\partial t$, where now ξ^a and τ may depend on v^a. Then

$$\begin{aligned} L_X(dx^a - v^a dt) &= d\xi^a - \eta^a dt - v^a d\tau \\ &= \left(\frac{\partial \xi^a}{\partial x^b} - v^a \frac{\partial \tau}{\partial x^b}\right)(dx^b - v^b dt) + \left(\frac{\partial \xi^a}{\partial v^b} - v^a \frac{\partial \tau}{\partial v^b}\right)(dv^b - \Lambda^b dt) \\ &\quad + [\Gamma(\xi^a) - v^a\Gamma(\tau) - \eta^a]\, dt \end{aligned}$$

So $L_X(dx^a - v^a dt) \in C$ if and only if

$$\frac{\partial \xi^a}{\partial v^b} - v^a \frac{\partial \tau}{\partial v^b} = 0$$

$$\eta^a = \Gamma(\xi^a) - v^a\Gamma(\tau)$$

In fact the first of these conditions allows one to write the second as

$$\eta^a = \left(v^b \frac{\partial \xi^a}{\partial x^b} + \frac{\partial \xi^a}{\partial t}\right) - v^a\left(v^b \frac{\partial \tau}{\partial x^b} + \frac{\partial \tau}{\partial t}\right)$$

as before.

I now assume that X preserves contact forms, and investigate the condition

$$X \lrcorner\, d\theta = dF + \alpha$$

where F is a constant of the motion, and $\alpha \in C$. Now

$$X \lrcorner\, d\theta = -g_{ab}(\xi^b - v^b\tau)(dv^a - \Lambda^a dt) + \textit{terms in } (dx^a - v^a dt)$$

while

$$dF = \frac{\partial F}{\partial x^a}(dx^a - v^a dt) + \frac{\partial F}{\partial v^a}(dv^a - \Lambda^a dt)$$

since $\Gamma(F) = 0$. Thus to satisfy the conditions, ξ^a and τ must be chosen to satisfy

$$g_{ab}(\xi^b - v^b\tau) = -\frac{\partial F}{\partial v^a}$$

This may be written

$$\xi^a - v^a\tau = -g^{ab}\frac{\partial F}{\partial v^b}$$

where g^{ab} denotes the inverse of g_{ab}, in the usual way. Now ξ^a and τ must also satisfy

$$\frac{\partial \xi^a}{\partial v^b} - v^a\frac{\partial \tau}{\partial v^b} = 0$$

But by differentiating the equation for $\xi^a - v^a\tau$, one obtains

$$\frac{\partial \xi^a}{\partial v^b} - v^a\frac{\partial \tau}{\partial v^b} - \tau\delta_b^a = -\frac{\partial}{\partial v^b}\left(g^{ac}\frac{\partial F}{\partial v^c}\right)$$

Thus it is necessary that

$$\frac{\partial}{\partial v^b}\left(g^{ac}\frac{\partial F}{\partial v^c}\right) = \tau\delta_b^a$$

that is, that $(\partial/\partial v^b)(g^{ac}\,\partial F/\partial v^c)$ be a multiple of the identity matrix, if the vector field X is to exist. There is no reason why this condition should hold, in general, for a constant of the motion; and in fact it is easy to find examples in which it does not hold. For example, in the case of the harmonic oscillator, $L = \frac{1}{2}\delta_{ab}(v^a v^b - w^2 x^a x^b)$, the function $F = \frac{1}{2}A_{ab}(v^a v^b + w^2 x^a x^b)$ is a constant of the motion for any constant symmetric matrix A_{ab}. But then

$$\frac{\partial v^b}{\partial}\left(g^{ac}\frac{\partial F}{\partial v^c}\right) = \delta^{ac}A_{bc}$$

which will be a multiple of the identity only if A_{ab} is itself a multiple of the identity. Thus the only one of these constants of the motion for which there is a corresponding vector field according to Noether's theorem is the energy.

Compared with the Cartan form approach, Noether's theorem has several drawbacks. First, the notion of a symmetry is by no means so clear cut. In the Cartan form approach, a symmetry is a transformation that preserves the fundamental geometrical object in the theory, and when properly normalized, permutes the orbits of dynamical systems. The one-form Ldt is not so directly related to the dynamics of the system as θ is, so preserving Ldt is not such a clear definition of symmetry as preserving $d\theta$ is. Secondly, in the Cartan version, there is a natural algebraic relationship between symmetries and constants of the motion, which is almost one-to-one when the symmetries are normalized (though two constants of the motion, which differ, trivially, by a number, give rise to the same symmetry vector field), and which is onto. This feature does not survive in the Noether's theorem approach. In fact, Noether's theorem when applied to the harmonic oscillator fails to produce what are in some ways the most interesting constants of the motion, those associated with the so-called hidden symmetries.

6. *Noether's Theorem and the Calculus of Variations*

It may be objected to the foregoing that the advantage of Noether's theorem is that it arises from the formulation of Lagrange's equations by means of a variational principle. In fact, all relationships between one-forms may be expressed in terms of integrals, so that this advantage, if such it is, holds for the Cartan form theory as well. I shall now briefly justify this remark.

It is convenient to discuss variation of integrals in some generality. Let ω be a one-form on a manifold N; γ a curve in N; and X a vector field on N, with one parameter group ψ_u. The curve $\psi_u \circ \gamma$ is the curve obtained from γ by transforming it by ψ_u. The one-form ω may be integrated along this curve, over some fixed interval of the parameter, say $[t_1, t_2]$. Let

$$I(u) = \int_{\psi_u \circ \gamma} \omega$$

The problem is to find an expression for $\dot{I}(0)$, the derivative of I with respect to u at $u = 0$, in terms of X and ω. Now

$$\int_{\psi_u \circ \gamma} \omega = \int_\gamma \psi_u^* \omega$$

from which it follows immediately that

$$\dot{I}(0) = \int_\gamma L_X \omega$$

Using the expression $L_X \omega = X \lrcorner\ d\omega + d\langle X, \omega\rangle$, one obtains

$$\dot{I}(0) = \int_\gamma X \lrcorner\ d\omega + \Big[\langle X, \omega \rangle \Big]_{t_1}^{t_2}$$

This is the fundamental variational formula in integral form.

To apply this result in the case of a dynamical system, one must choose an appropriate one-form for ω. The available choices are θ, the Cartan form, and

Ldt; and clearly θ is the better choice because when γ is an integral curve of Γ, the integral $\int_\gamma X \lrcorner\, d\theta$ vanishes since Γ is characteristic for $d\theta$. If $L_X\theta$ is exact, or differs from an exact form df by a contact form, then

$$\int_\gamma L_X\theta = [f]_{t_1}^{t_2}$$

and so $f - \langle X, \theta \rangle$ is constant along γ.

Clearly this integral version adds nothing to the previous discussion.

7. *Conclusion*

I hope I have demonstrated the superiority of the Cartan form description of the relation between symmetries and constants of the motion over the Noether's theorem approach; and also the superiority of modern differential geometry techniques over classical ones for clarity and economy in deriving results in Lagrangian mechanics.

References

Hermann, R. (1968). *Differential Geometry and the Calculus of Variations* (particularly Chap. 15). Academic Press, New York.

Lovelock, D., and Rund, H. (1975). *Tensors, Differential Forms and Variational Principles,* Chap. 6. John Wiley, New York.

Palmieri, C., and Vitale, B. (1970). *Nuovo Cimento*, **66A**, 299.

Saletan, E. J., and Cromer, A. H. (1971). *Theoretical Mechanics*, Chap. III. John Wiley, New York.